Assignment 7

Exercise 1

Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ and define the process $(M_t)_{t\geq 0}$ by $M_t := \sup_{0\leq s\leq t} B_s$. Consider the random variable

$$D := \sup_{0\leq s\leq 1}\bigg\{\sup_{0\leq t\leq s}\{B_t - B_s\}\bigg\}.$$

That is, D characterises the maximal possible 'downfall' in trajectories of the Brownian motion on the time interval $[0,1]$.

1) Show that $D \overset{\text{law}}{=} \sup_{0\leq t\leq 1}|B_t|$.

   Hint: you can use (and prove if you want!) Lévy's theorem, which states that the processes $M-B$ and $|B|$ have the saw law under $\mathbb{P}$.

2) Show that $\sup_{0\leq t\leq 1}|B_t| \overset{\text{law}}{=} 1/\sqrt{\bar{T}_1}$, where $\bar{T}_1 := \inf\{t>0 : |B_t|\geq 1\}$.

3) Conclude that $\mathbb{E}^{\mathbb{P}}[D] = \sqrt{\pi/2}$.

Exercise 2

Fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let B be a Brownian motion in $\mathbb{R}^d$ (with respect to its $\mathbb{P}$-completed natural filtration) for some integer $d\geq 2$. For any $x\in\mathbb{R}^d$, we let $B^x := x + B$, and for any $x\in\mathbb{R}^d\setminus\{0\}$, and any $0<a<\|x\|<b$, we let

$$\tau_a := \inf\{t\geq 0 : \|B^x_t\|\leq a\},\ \tau_b := \inf\{t\geq 0 : \|B^x_t\|\geq b\}.$$

1) Assume $d\geq 3$ and show that $X^x_t := \big\|B^x_{\tau_a\wedge t}\big\|^{2-d}$, $t\geq 0$, is a bounded $(\mathbb{F},\mathbb{P})$-martingale.

2) Assume that $d=2$, and show that $Y^x_t := -\log\big(\big\|B^x_{\tau_a\wedge\tau_b\wedge t}\big\|\big)$, $t\geq 0$, is a bounded $(\mathbb{F},\mathbb{P})$-martingale.

3) Show that for any $x\in\mathbb{R}^d\setminus\{0\}$, $\mathbb{P}\big[B^x_t\neq 0,\ \forall t\geq 0\big] = 1$.

4) Assume $d\geq 3$, and show that for any $x\in\mathbb{R}^d$, $\mathbb{P}\big[\lim_{t\to+\infty}\|B^x_t\| = +\infty\big] = 1$.

Exercise 3

Let B be a Brownian motion in $\mathbb{R}^3$, $0\neq x\in\mathbb{R}^3$ and define the process $M=(M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{\|x+B_t\|}.$$

This is well defined as a 3-dimensional Brownian motion does not hit points, as seen in the previous exercise.

1) Show that M is a continuous local martingale. Moreover, show that M is bounded in $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$, that is

$$\sup_{t\geq 0}\mathbb{E}^{\mathbb{P}}[|M_t|^2] < +\infty.$$

2) Show that M is a *strict local martingale*, i.e., M is not a martingale.

   Hint: Show that $\mathbb{E}^{\mathbb{P}}[M_t] \to 0$ as $t \to +\infty$. To this end, similarly to 1), compute $\mathbb{E}^{\mathbb{P}}[M_t]$ and use the reverse triangle inequality as a first estimate. Then compute the resulting integral using spherical coordinates.

Exercise 4

Let B be a Brownian motion. For all $y \in \mathbb{R}_+^\star$, we define

$$T_y := \inf\big\{t \geq 0 : B_t \geq y\big\}.$$

Fix $a > 0$ and $b > 0$ and define

$$T_{a,b} := T_{-a} \wedge T_b.$$

1) Justify that $T_{a,b}$ is an $\mathbb{F}^{B,\mathbb{P}}$–stopping time.

2) Fix $\theta \in \mathbb{R}$ and define $X_t^{\theta,a}$ by

$$X_t^{\theta,a} := \sinh(\theta(B_t + a)) \exp\bigg(-\frac{\theta^2}{2}t\bigg).$$

   Show that $X^{\theta,a}$ is an $(\mathbb{F}^{B,\mathbb{P}}, \mathbb{P})$-martingale.

3) Deduce that

$$\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\frac{\theta^2}{2}T_b\bigg)\mathbf{1}_{\{T_b < T_{-a}\}}\bigg] = \frac{\sinh(\theta a)}{\sinh(\theta(a+b))},$$

   and then that

$$\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\frac{\theta^2}{2}T_{-a}\bigg)\mathbf{1}_{\{T_b > T_{-a}\}}\bigg] = \frac{\sinh(\theta b)}{\sinh(\theta(a+b))},$$

   and finally that

$$\mathbb{E}\bigg[\exp\bigg(-\frac{\theta^2}{2}T_{a,b}\bigg)\bigg] = \frac{\cosh\big(\frac{\theta(a-b)}{2}\big)}{\cosh\big(\frac{\theta(a+b)}{2}\big)}.$$

4) Deduce

$$\mathbb{P}\big[T_b < T_{-a}\big] = \frac{a}{a+b},\ \mathbb{P}\big[T_b > T_{-a}\big] = \frac{b}{a+b},$$

   and then that the random variable $\sup_{0 \leq t \leq T_{-1}} B_t$ has the same law as $(1-U)/U$ where U is uniform on $[0,1]$.